Interlacing properties and the Schur-Szegő composition

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Abstract

Each degree n polynomial in one variable of the form $(x+1)(x^{n-1}+c_1x^{n-2}+\cdots+c_{n-1})$ is representable in a unique way as a Schur-Szegő composition of $n-1$ polynomials of the form $(x+1)^{n-1}(x+a_i)$, see [5], [2] and [7]. Set $\sigma_j := \sum_{1\leq i_1<\cdots<i_j\leq n-1} a_{i_1}\cdots a_{i_j}$. The eigenvalues of the affine mapping $(c_1,\ldots,c_{n-1})\mapsto(\sigma_1,\ldots,\sigma_{n-1})$ are positive rational numbers and its eigenvectors are defined by hyperbolic polynomials (i.e. with real roots only). In the present paper we prove interlacing properties of the roots of these polynomials.

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1 The Schur-Szegő composition and the mapping Φ

The present paper deals with the *composition of Schur-Szegő (CSS)* of degree n polynomials in one variable. For the polynomials $P := \sum_{j=0}^n a_jx^j$ and $Q := \sum_{j=0}^n b_jx^j$ their CSS is defined by the formula $P*Q := \sum_{j=0}^n a_jb_jx^j/C_n^j$ ($C_n^j = n!/j!(n-j)!$, $a_i, b_i \in \mathbf{C}$). In the monographies [12] and [13] one can find properties and applications of the CSS. The CSS is associative, commutative and polylinear. It can be defined for more than two polynomials by the formula

$$P_1*\cdots*P_s = \sum_{j=0}^n a_j^1\cdots a_j^s x^j/(C_n^j)^{s-1} \quad \text{where} \quad P_i = \sum_{j=0}^n a_j^i x^j \ .$$

The role of the unity in the CSS is played by the polynomial $(x+1)^n$ in the sense that for every degree n polynomial P one has $P*(x+1)^n = P$.

Definition 1 A real polynomial is *hyperbolic* (resp. *strictly hyperbolic*) if it has only real (resp. only real and distinct) roots.

In papers [10], [5] and [6] the question is considered how many of the roots of the CSS of two hyperbolic or only real polynomials are real negative, zero or positive. In the case of hyperbolic polynomials the exhaustive answer is given in [5], while [6] contains sufficient conditions for the realizability of certain cases defined by the number of negative, positive and complex roots of P, Q and $P*Q$. In the case of two hyperbolic polynomials P and Q one of which has only negative roots, the multiplicity vector of the roots of $P*Q$ is completely defined by the ones of P and Q, see [10].

Call *composition factor* any polynomial of the form $K_a := (x+1)^{n-1}(x+a)$. In paper [5] it is announced and in paper [2] it is proved that any monic degree n complex polynomial P such that $P(-1)=0$ is representable in the form

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$$P = K_{a_1} * \cdots * K_{a_{n-1}} \tag{1}$$

where the numbers $a_i \in \mathbf{C}$ are unique up to permutation. To extend this presentation to the case when P is not necessarily monic and, more generally, to the case when $\deg P \leq n$, one has to admit the presence of composition factors $K_\infty := (x+1)^{n-1}$ and of a constant factor in the right-hand side of (1). When the polynomial P is real, then part of the numbers a_i are real while the rest form complex conjugate couples. (Otherwise conjugation of (1) defines a new $(n-1)$-tuple of numbers a_i.)

Denote by σ_j the jth elementary symmetric polynomial of the numbers a_i, i.e. $\sigma_j := \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} a_{i_1} \cdots a_{i_j}$. Set $P := (x+1)(x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-1})$. Consider the mapping

$$\Phi \ : \ (c_1, \ldots, c_{n-1}) \ \mapsto \ (\sigma_1, \ldots, \sigma_{n-1}) \ .$$

In paper [7] it is shown that this mapping is affine, that its eigenvalues are the positive rational numbers listed in the first line of the following displayed formula, and the corresponding eigenvectors are defined by the polynomials of its second line:

$$\begin{array}{cccccc} \lambda_1 = 1 & \lambda_2 = \frac{n}{(n-1)} & \lambda_3 = \frac{n^2}{(n-1)(n-2)} & \lambda_4 = \frac{n^3}{(n-1)(n-2)(n-3)} & \ldots & \lambda_{n-1} = \frac{n^{n-2}}{(n-1)!} \\ (x+1)^{n-1} & x(x+1)^{n-2} & x(x+1)^{n-3}Q_1(x) & x(x+1)^{n-4}Q_2(x) & \ldots & x(x+1)Q_{n-3}(x) \end{array}$$

Remark 2 1) The polynomial Q_j is degree j, monic, strictly hyperbolic, with all roots positive and *self-reciprocal*, i.e. $x^j Q_j(1/x) = \pm Q_j(x)$. In particular, $Q_1 = x - 1$. Hence if x_0 is root of Q_j, then $1/x_0$ is also such a root. If j is odd, then $Q_j(1) = 0$.

2) In the above list the "eigenpolynomials" are all of degree $n-1$ and divisible by $(x+1)$. If one considers Φ as a linear (not affine) mapping $(c_0, \ldots, c_{n-1}) \mapsto (\sigma_0, \ldots, \sigma_{n-1})$, i.e. when $P = (x+1)(c_0 x^{n-1} + c_1 x_{n-2} + \cdots + c_{n-1})$, then one sets $\sigma_0 = c_0$ and has to add a second eigenvalue 1 to which there corresponds the eigenpolynomial $(x+1)^n$. To the eigenvalue 1 there correspond two Jordan blocks of size 1.

3) It is shown in [9] that for j fixed and when $n \to \infty$, the polynomial $xQ_j(-x)$ tends to the *Narayana polynomial* $\sum_{i=1}^{n} N_{n,i} x^i$ where $N_{n,i}$ are the *Narayana numbers* $C_n^i C_n^{i-1}/n$.

Denote by $0 < x_1 < \cdots < x_j$ the roots of Q_j, by $0 < y_1 < \cdots < y_{j+1}$ the ones of Q_{j+1} and by $0 < z_1 < \cdots < z_{j+2}$ the ones of Q_{j+2}. In the present paper we prove the following two theorems. In their proofs we use some known results about hyperbolic polynomials and the mapping Φ, see Section 2.

Theorem 3 *(First interlacing property) The roots of Q_j and Q_{j+1} interlace ($j = 1, \ldots, n-4$), i.e. $y_1 < x_1 < y_2 < x_2 < \cdots < y_j < x_j < y_{j+1}$.*

The theorem is proved in Section 3.

Theorem 4 *(Second interlacing property) For $i = 1, \ldots, [j/2]$ one has $x_i \in (z_i, z_{i+1})$ and $x_{j+1-i} \in (z_{j+2-i}, z_{j+3-i})$. If j is odd, then $x_{(j+1)/2} = z_{(j+3)/2} = 1$.*

The theorem is proved in Section 4. An analog of Theorem 4 holds for the roots of Gegenbauer polynomials and their derivatives and for the roots of Narayana polynomials, see Section 5.

2 Preliminaries

In the present section we recall some classical properties of hyperbolic polynomials and some properties of the mapping Φ. The following theorem is a well-known result which is used in the proofs (see Section 5 in [11]):

Theorem 5 *Suppose that P and Q are two polynomials with no root in common. Then they are hyperbolic and their roots interlace if and only if for any $(\theta, \mu) \in (\mathbf{R}^2 \backslash \{(0,0)\})$ the polynomial $\theta P + \mu Q$ has only real roots.*

Next, we recall some nontrivial properties of the mapping Φ:

Proposition 6 *If the degree n polynomials P and Q have nonzero roots x_P, x_Q of multiplicities m_P, m_Q such that $m_P + m_Q \geq n$, then $-x_P x_Q$ is a root of $P * Q$ of multiplicity $m_P + m_Q - n$.*

This is Proposition 1.4 in [10] (in [10] the condition the roots to be nonzero is omitted which is not correct).

Remark 7 In the right-hand side of (1) there are exactly l composition factors with $a_i \neq -1$ if and only if the multiplicity of (-1) as a root of P equals $n - l$. This follows from Proposition 6 applied to the right-hand side $l - 1$ times, the roles of x_P, x_Q being played by the roots equal to (-1).

Notation 8 For $i = 0, 1, \ldots, n-1$ we set $b_i := -i/(n-i)$. Set $b_n := -\infty$.

Proposition 9 *1) Suppose that $P = x^m R$, $degR \leq n - m$. Then m of the numbers a_i equal (up to permutation) b_0, b_1, …, b_{m-1}.*

2) Suppose that $degP = n - m$. Then m of the numbers a_i equal (up to permutation) b_n, b_{n-1}, …, b_{n-m+1}.

The proposition coincides with part 3) of Remark 2 in [7].

Proposition 10 *For $a < 0$ there is exactly one change of sign in the sequence of coefficients of the polynomial K_a. For $a > 0$ there is no change of sign in it.*

Indeed, one has

$$K_a = (x+1)^{n-1}(x+a) = \sum_{j=0}^{n} (aC_{n-1}^j + C_{n-1}^{j-1})x^j = x^n + \sum_{j=0}^{n-1} C_{n-1}^j \left(a + \frac{j}{n-j}\right) x^j$$

and the signs of the sequence of coefficients are the same as the ones of the sequence a, $a + 1/(n-1)$, $a + 2/(n-2)$, …, $a + n - 1$, 1. $\square$

Proposition 11 *If the hyperbolic polynomial P has l positive roots, then at least l of the numbers a_i are negative and distinct.*

The proposition follows from the final remark in [8].

Proposition 12 *The mapping Φ preserves self-reciprocity.*

This follows from the fact that the matrix of Φ (considered as a linear, not affine mapping, see part 2) of Remark 2) is centre-symmetric, see Proposition 10 in [7].

3 Proof of Theorem 3

1^0. Consider two polynomials of the form $U := x(x+1)^{n-2-j}T_j(x)$ and $V := x(x+1)^{n-3-j}T_{j+1}(x)$ where the polynomials T_i are monic, hyperbolic and self-reciprocal, $\deg T_i = i$. Assume also that the roots of T_j and T_{j+1} are all distinct, positive and that they interlace (in the same sense as in the claim of Theorem 3 about the roots of Q_j and Q_{j+1}).

Proposition 13 *Under these assumptions, for every $(\theta,\mu) \in (\mathbf{R}^2 \backslash \{(0,0)\})$ the polynomial $T := (x+1)\theta T_j + \mu T_{j+1}$ has all roots real and distinct at least j of which are positive. The last root might be positive, negative or 0 (in the last case the polynomial $\theta U + \mu V = x(x+1)^{n-3-j}T$ has a double root at 0); it might equal (-1) in which case the polynomial $\theta U + \mu V$ has an $(n-2-j)$-fold root at (-1).*

Proof:

The cases $\theta = 0 \neq \mu$ and $\theta \neq 0 = \mu$ are self-evident, so suppose that $\theta > 0$, $\mu \neq 0$. The roots of the polynomials $(x+1)T_j$ and T_{j+1} interlace. Hence in every interval between two consecutive roots of T_{j+1} the polynomial T takes at least one positive and at least one negative value.

Indeed, it suffices to prove this for the interval between the two largest roots $0 < a < b$ of T_{j+1}. Denote by c the root of T_j belonging to (a,b). If $\mu > 0$, then $T(b) > 0$, $T(c) < 0$. If $\mu < 0$, then $T(c) > 0$, $T(a) < 0$.

Hence in all cases the polynomial T has a root in (a,b). In the same way one shows that it has a root between any two consecutive roots of T_{j+1}, i.e. it has at least j positive distinct roots. For $(\theta,\mu) = (0,1)$ its last root is positive, for $(\theta,\mu) = (1,0)$ it equals (-1). It is clear that by arguments of continuity there exists $(\theta,\mu) \neq (0,0)$ for which the last root equals 0. □

Proposition 14 *For U and V as above and for every $(\theta,\mu) \in (\mathbf{R}^2 \backslash \{(0,0)\})$ the polynomial $\Phi(\theta U + \mu V)$ is hyperbolic. It has a simple or double root at 0, a root at (-1) of multiplicity $(n-j-3)$ or $(n-j-2)$ and at least j distinct positive roots.*

Indeed, by Proposition 13 the polynomial $\theta U + \mu V$ is hyperbolic and has at least j distinct positive roots. By Remark 7 it has an $(n-j-3)$- or $(n-j-2)$-fold root at (-1).

By part 1) of Proposition 9 the polynomial $\Phi(\theta U + \mu V)$ has a root at 0; by Proposition 11 it has j or $j+1$ distinct positive roots. Hence in all cases it is hyperbolic. Indeed, it is of degree $n-1$, and except the positive distinct roots and the ones at 0 and (-1), there remains only one root not accounted for which is necessarily real. □

2^0. The above proposition and Theorem 5 imply that the roots of the polynomials $\Phi(U)/x(x+1)^{n-3-j}$ and $\Phi(V)/x(x+1)^{n-3-j}$ are real and interlace. Using the same type of arguments one sees that in the special cases $(\theta,\mu) = (1,0)$ or $(0,1)$ the multiplicity of the root at (-1) of $\Phi(\theta U + \mu V)$ equals respectively $(n-2-j)$ and $(n-3-j)$. In both cases the root at 0 is simple. Hence there are respectively j and $j+1$ distinct positive roots.

3^0. This means that the polynomials $\Phi(U)$, $\Phi(V)$ satisfy the conditions of Proposition 14. Hence the conclusion of the proposition applies also to the polynomial $\Phi^2(\theta U + \mu V)$ and in the same way to $\Phi^k(\theta U + \mu V)$ for $k \in \mathbf{N}$, i.e. we have

Corollary 15 *For $k \in \mathbf{N}$, for U and V as above and for every $(\theta,\mu) \in (\mathbf{R}^2 \backslash \{(0,0)\})$ the polynomial $\Phi^k(\theta U + \mu V)$ is hyperbolic. It has a simple or double root at 0, a root at (-1) of multiplicity $(n-j-3)$ or $(n-j-2)$ and at least j distinct positive roots.*

Proposition 16 *All polynomials $\Phi^k(U)$ and $\Phi^k(V)$ are self-reciprocal. The set of roots of the polynomial $\Phi^k(U)$ (resp. $\Phi^k(V)$) tends for $k \to \infty$ to the one of $x(x+1)^{n-2-j}Q_j(x)$ (resp. the one of $x(x+1)^{n-3-j}Q_{j+1}(x)$).*

Proof:

Self-reciprocity of $\Phi^k(U)$ and $\Phi^k(V)$ follows from Proposition 12. Hence about half of the roots of the polynomials $\Phi^k(U)/x(x+1)^{n-2-j}$ and $\Phi^k(V)/x(x+1)^{n-3-j}$ belong to the interval $(0,1)$.

Set $W_j := x(x+1)^{n-2-j}Q_j$. The polynomials W_i, $i = 0, \ldots, j$, are a basis of the linear space of degree $\leq n-1$ polynomials divisible by $x(x+1)^{n-2-j}$. Set $U := \sum_{i=0}^{j} \alpha_i W_i$, $V := \sum_{i=0}^{j+1} \beta_i W_i$. One has $\alpha_j \neq 0 \neq \beta_{j+1}$ because otherwise U (resp. V) is divisible by $(x+1)^{n-1-j}$ (resp. by $(x+1)^{n-2-j}$).

Hence $\Phi^k(U) = \sum_{i=0}^{j} \alpha_i \lambda_i^k W_i$, $\Phi^k(V) = \sum_{i=0}^{j+1} \alpha_i \lambda_i^k W_i$, and as $\lambda_i > 0$ and $\lambda_i < \lambda_{i+1}$, the set of roots of $\Phi^k(U)$ (resp. of $\Phi^k(V)$) tends for $k \to \infty$ to the one of W_j (resp. the one of W_{j+1}). $\square$

4^0. The numbers of positive roots of $\Phi^k(U)$ and $\Phi^k(V)$ are respectively j and $j+1$. These roots interlace. This follows from Corollary 15 and from Theorem 5. The above proposition implies that

$$y_1 \leq x_1 \leq y_2 \leq x_2 \leq \cdots \leq y_j \leq x_j \leq y_{j+1} \tag{2}$$

The roots of Q_j and the ones of Q_{j+1} being distinct, there can be no two consecutive equalities in this string of inequalities.

5^0. Suppose that there is an equality of the form $x_i = y_i$ or $x_i = y_{i+1}$. Denote by x_0 a root common for Q_j and Q_{j+1}. The roots of Q_j and Q_{j+1} being simple one can find $(\theta_0, \mu_0) \in (\mathbf{R}^2 \backslash \{(0,0)\})$ such that the polynomial $L := \theta_0 W_j + \mu_0 W_{j+1}$ has a multiple root at x_0.

Consider the polynomial $M := (\theta_0/\lambda_j)W_j + (\mu_0/\lambda_{j+1})W_{j+1}$. One has $L = \Phi(M)$. The polynomial M can be considered as a limit of a sequence of polynomials of the form $M_s := (\theta_0/\lambda_j)U_s + (\mu_0/\lambda_{j+1})W_{j+1}$, $s = 1, 2, \ldots$, where $U_s = x(x+1)^{n-2-j}T_{j,s}$ the roots of $T_{j,s}$ being positive, distinct and interlacing with the ones of Q_{j+1}. For each s fixed the polynomial M_s has at least j positive distinct roots (and by Proposition 11 this is also the case of $L_s := \Phi(M_s)$), hence M and L have each at least j positive roots counted with multiplicity.

In the right-hand side of equality (1) with $P = M$ there are $n - j - 3$ composition factors with $a_i = 1$ which can be skipped. From the remaining ones there are j or $j+1$ with $a_i < 0$ and one or two with $a_i = 0$. If there are (at least) two composition factors with the same $a_i < 0$ (which is the case when $L = \Phi(M)$ has a multiple positive root), then their composition is a polynomial with all coefficients nonnegative, i.e. without change of the sign in the sequence of the coefficients. Hence the remaining composition factors with $a_i < 0$ are not more than $j-1$ and their composition can be a polynomial with not more than $j-1$ changes of the sign in the sequence of the coefficients (see Proposition 10). The composition factor(s) K_0 add(s) no changes of sign in this sequence.

But then according to the Descartes rule the polynomial M must have not more than $j-1$ positive roots counted with multiplicity which is a contradiction. $\square$

4 Proof of Theorem 4

1^0. Recall that $W_\nu = x(x+1)^{n-2-\nu}Q_\nu$. Denote by B a polynomial of the form $x(x+1)^{n-2-j}C$ where the polynomial C is monic, degree j, self-reciprocal and has a root in each of the intervals

(z_i, z_{i+1}) and (z_{j+2-i}, z_{j+3-i}) for $i = 1, \ldots, [j/2]$. (If j is odd, then $C(1) = 0$.) In this sense we say that the roots of the polynomials C and Q_{j+2} satisfy the second interlacing property.

Remark 17 The polynomial $\Phi(B)$ is divisible by $x(x+1)^{n-2-j}$. This follows from part 1) of Proposition 9 and from Remark 7. In the same way for any $k \in \mathbf{N}$ one can conclude that the polynomial $\Phi^k(B)$ is divisible by $x(x+1)^{n-2-j}$.

Lemma 18 *One can choose the polynomial C such that for any $k \in \mathbf{N} \cup \{0\}$ the polynomials Q_{j+2} and $\Phi^k(B)/(x(x+1)^{n-2-j})$ have no root different from 1 in common.*

Proof:

Suppose that j is even. Then 1 is not a root of C and not a root of Q_{j+2}. Fix $k \in \mathbf{N}$. The condition the polynomials Q_{j+2} and $\Phi^k(B)/(x(x+1)^{n-2-j})$ to have a root in common reads $\Delta_k :=\mathrm{Res}(Q_{j+2}, \Phi^k(B)/(x(x+1)^{n-2-j})) = 0$. This equality defines a proper algebraic subvariety in the space of the half of the coefficients of the polynomial C; half of them because C is self-reciprocal.

It is clear that Δ_k is a not identically constant polynomial in the half of the coefficients of C. Indeed,

1) for $k = 0$ this results from the fact that if one fixes all roots of C except x_0 and $1/x_0$ the remaining ones being nonroots of Q_{j+2}, then Δ_0 will equal 0 precisely when x_0 and $1/x_0$ are roots of Q_{j+2};

2) for $k > 0$ this follows from 1) and from the mapping Φ^k being nondegenerate and preserving self-reciprocity, see [7].

It is reasonable to consider the space $\tilde{R}$ of the roots of C which belong to $(0, 1)$ and not the space of all its roots. Indeed, the roots of C belonging to $(1, \infty)$ equal $1/x_*$ where x_* is a root in $(0, 1)$; when deg$C$ is odd, then C has a simple root at (-1).

The roots of C are real and distinct. Therefore the following statement is true (see its proof at the end of the proof of the lemma):

Locally the mapping " U : roots of C belonging to $(0, 1) \mapsto$ first half of its coefficients " is a diffeomorphism.

Thus the condition $\Delta_k = 0$ defines a proper algebraic subvariety in the space $\tilde{R}$. Its complement Z_k is a Zariski open dense subset of $\tilde{R}$. The intersection $\cap_{k=0}^{\infty} Z_k$ is nonempty. To choose C as claimed by the lemma is the same as to choose the roots of C which belong to $(0, 1)$ from the set $\cap_{k=0}^{\infty} Z_k$.

If j is odd, then 1 is a simple root of C and of Q_{j+2} and one can apply the same reasoning to the polynomials $C/(x-1)$ and $Q_{j+2}/(x-1)$.

Prove the above statement. For each two roots of C of the form x_0, $1/x_0$ consider the product $(x - x_0)(x - 1/x_0) = x^2 - sx + 1$. Write down C as a product of such degree 2 polynomials. The respective quantities s are all real and distinct. Therefore the mapping Y which sends their tuple into the tuple of the values of their elementary symmetric polynomials is a diffeomorphism. The mapping T sending these symmetric polynomials into the tuple of the first half of the coefficients of C is affine nondegenerate triangular (to be checked directly). The mapping V sending the roots of C belonging to $(0, 1)$ into the quantities s is a diffeomorphism (easy to check). The mapping U equals $T \circ Y \circ V$. Hence it is a local diffeomorphism. $\square$

2^0. From now till the end of the proof of Theorem 4 we assume that the roots of C satisfy the conclusion of Lemma 18.

Proposition 19 *The polynomial $\Phi(B)$ is self-reciprocal. The roots of $\Phi(B)/(x(x+1)^{n-2-j})$ and Q_{j+2} satisfy the second interlacing property.*

Before proving the proposition deduce Theorem 4 from it. For every $k \in \mathbf{N}$ the polynomial $\Phi^k(B)$ is divisible by $x(x+1)^{n-2-j}$, see Remark 17. Applying k times Proposition 19 one shows that for any $k \in \mathbf{N}$ the polynomial $\Phi^k(B)/(x(x+1)^{n-2-j})$ is self-reciprocal and the roots of $\Phi^k(B)/(x(x+1)^{n-2-j})$ and Q_{j+2} satisfy the second interlacing property.

3^0. Set $Q_0 := 1$. The polynomial B can be presented in a unique way in the form $\sum_{\nu=0}^{j} \gamma_\nu W_\nu$, $\gamma_\nu \in \mathbf{R}$. Indeed, a priori it can be presented in the form $a(x+1)^{n-1} + \sum_{\nu=0}^{n-1} \gamma_\nu W_\nu$, see the eigenvectors of the mapping Φ before Remark 2. It is divisible by x and by $(x+1)^{n-2-j}$. Hence $a = \gamma_{j+1} = \gamma_{j+2} = \cdots = \gamma_{n-1} = 0$. Moreover, $\gamma_j \neq 0$, otherwise B must be divisible by $(x+1)^{n-1-j}$ which is false.

One has $\Phi^k(B) = \sum_{\nu=0}^{j} (\lambda_\nu)^k \gamma_\nu W_\nu = (\lambda_j)^k \sum_{\nu=0}^{j} (\lambda_\nu/\lambda_j)^k \gamma_\nu W_\nu$. As $1 \leq \lambda_\nu < \lambda_j$, for large values of k the positive roots of the right-hand side are close to the ones of the polynomial $(\lambda_j)^k \gamma_j Q_j$. Passing to the limit when $k \to \infty$ one sees that the polynomial Q_j has a root in each of the intervals $[z_i, z_{i+1}]$ and $[z_{j+2-i}, z_{j+3-i}]$ for $i = 1, \ldots, [j/2]$.

It is clear that for j odd one has $Q_j(1) = 0$. Therefore the second interlacing property will be proved if one manages to exclude the possibility Q_j and Q_{j+2} to have a common positive root different from 1. We do this by analogy with 4^0 and 5^0 of the proof of Theorem 3.

4^0. Suppose that such a root x_0 exists. Then $1/x_0$ is also such a root.

Consider the polynomials W_j and W_{j+2} as degree n, not $n-1$ ones. They are self-reciprocal. One has simultaneously $x^n W_j(1/x) = W_j(x)$, $x^n W_{j+2}(1/x) = W_{j+2}(x)$ or $x^n W_j(1/x) = -W_j(x)$, $x^n W_{j+2}(1/x) = -W_{j+2}(x)$. Indeed, the parities of the multiplicities of their roots 1 and (-1) are the same. Therefore any linear combination $\theta W_j + \mu W_{j+2}$ is a self-reciprocal polynomial.

The root x_0 (considered as a root of W_j or W_{j+2}) is simple. Therefore there exists a couple $(\theta_0, \mu_0) \in (\mathbf{R}^2 \backslash \{(0,0)\})$ such that the polynomial $W := \theta_0 W_j + \mu_0 W_{j+2}$ has a multiple root at x_0. By self-reciprocity it has a multiple root at $1/x_0$ as well.

5^0. The polynomial $H := (\theta_0/\lambda_j)W_j + (\mu_0/\lambda_{j+1})W_{j+2}$ can be presented as a limit of a sequence of polynomials of the form $H_s := (\theta_0/\lambda_j)G_s + (\mu_0/\lambda_{j+1})W_{j+2}$, $s = 1, 2, \ldots$, where $G_s = x(x+1)^{n-2-j} T_{j,s}$, $\deg T_{j,s} = j$, $T_{j,s}$ is self-reciprocal, its roots are positive, distinct and interlacing with the ones of Q_{j+2} in the sense of the second interlacing property. For each s fixed the polynomial H_s has at least j positive distinct roots (and by Proposition 11 this is true also for the polynomial $W_s := \Phi(H_s)$), hence H and $W = \Phi(H)$ have each at least j positive roots counted with multiplicity.

In the right-hand side of equality (1) with $P = H$ there are (except the $n-j-3$ composition factors K_1), j or $j+1$ composition factors with $a_i < 0$. There are among them two couples of equal factors. The composition of such a couple is a polynomial with all coefficients positive. Hence, it adds no change of the sign in the sequence of coefficients of the polynomial W. The remaining $\leq j-3$ composition factors with $a_i < 0$ bring not more than $j-3$ such changes and the factor(s) K_0 bring(s) no change at all. By the Descartes rule the polynomial H must have not more than $j-3$ positive roots counted with multiplicity which is a contradiction. $\square$

Proof of Proposition 19:

1^0. The polynomial $\Phi(B)$ is self-reciprocal because the matrix of the mapping Φ in the standard monomial basis is centre-symmetric, see Proposition 10 in [7].

2^0. Consider any linear combination $\Xi := \theta(x+1)^2 C + \mu Q_{j+2}$, $(\theta, \mu) \in (\mathbf{R}^2 \backslash \{(0,0)\})$. This degree $j+2$ polynomial has $j+2$ (resp. j) positive distinct roots for $\theta = 0$ (resp. for $\mu = 0$).

For $\theta \neq 0 \neq \mu$ it changes sign at any two consecutive positive roots of Q_{j+2} which are both smaller or both greater than 1. If j is odd, then $\Xi(1) = 0$. Hence for all (θ, μ), the polynomial Ξ has at least j real positive distinct roots (and at most one complex conjugate couple).

3^0. Hence the polynomial

$$\tilde{B} := \Phi(x(x+1)^{n-j-4}\Xi) = \Phi(\theta B + \mu W_{j+2}) = \theta\Phi(B) + \mu\lambda_{j+4}W_{j+2}$$

has a root at 0 (part 1) of Proposition 9), at least j distinct positive roots (Proposition 11), a root at (-1) of multiplicity at least $n-j-4$ (Remark 7). The remaining two roots are either complex or real; in the latter case one or both of them can coincide with some of the other roots. In particular, they can equal (-1).

4^0. Consider the rational function $g := \Phi(B)/W_{j+2}$. Lemma 18 allows to suppose that $\Phi(B)$ and W_{j+2} have no positive root different from 1 in common. Then $g(x)$ has no critical point for $x \in (0,1) \cup (1,\infty)$. Indeed, self-reciprocity of $\Phi(B)$ and W_{j+2} would imply that there are in fact two different such critical points, at x_0 and $1/x_0$, with equal critical values. If they are the only ones at their level set and if they are Morse ones, then for two different values of (θ, μ) the number of distinct positive roots of the polynomial $\tilde{B}$ changes by 4 which by 2^0 is impossible because the real positive roots of $\tilde{B}$ are not more than $j+2$ and not less than j.

If the critical points are either more than two on their level set or they are not Morse ones, then for some (θ, μ) the number of distinct positive roots of the polynomial $\tilde{B}$ is smaller than j – a contradiction with 2^0 again.

5^0. Therefore g must be monotonous between each two consecutive positive roots of Q_{j+2} which are smaller than 1, and between each two consecutive positive roots of Q_{j+2} which are greater than 1. The function g has a single critical point, at 1, which is a Morse one. This implies that the roots of $\Phi(B)/(x(x+1)^{n-2-j})$ (which are the level set $\{g = 0\}$ without the double root at (-1)) and the ones of Q_{j+2} satisfy the second interlacing property. $\square$

5 On Gegenbauer and Narayana polynomials

In this paper we define *the Gegenbauer polynomial* G_n of degree n ($n \geq 3$) as a polynomial in one variable which is divisible by its second derivative and whose first three coefficients equal $1, 0, -1$. It is easy to show that these conditions define a unique polynomial which is strictly hyperbolic and which is even (resp. odd) when n is even (resp. odd). Hence its nonconstant derivatives which are polynomials of even (resp. odd) degree are even (resp. odd) polynomials. The general definition of Gegenbauer polynomials $C_n^{(\lambda)}$ is different from the above one and depends on a parameter λ. Our definition corresponds to the particular case $\lambda = -1/2$. (For more details about Gegenbauer polynomials see Chapter 22 in [1].) Up to a rescaling and a nonzero constant factor the polynomial G_n' is the Legendre polynomial of degree $n-1$.

Denote by $\zeta_1 < \cdots < \zeta_l < 0$ the negative roots of $G_n^{(k)}$ where $l = [(n-k)/2]$. Denote by $\mu_1 < \cdots < \mu_{l-1} < 0$ the negative roots of $G_n^{(k+2)}$.

Theorem 20 *One has $\mu_i \in (\zeta_i, \zeta_{i+1})$, $i = 1, \ldots, l-1$.*

Proof:

The polynomial G_n satisfies the condition $G_n = (x^2 - a^2)G_n''$ where $\pm a$ are its roots with greatest absolute values. Differentiate this equality k times using the Leibniz rule:

$$G_n^{(k)} = (x^2 - a^2)G_n^{(k+2)} + 2kxG_n^{(k+1)} + k(k-1)G_n^{(k)} \ .$$

Hence $(\zeta_i^2 - a^2)G_n^{(k+2)}(\zeta_i) + 2k\zeta_i G_n^{(k+1)}(\zeta_i) = 0$. Recall that $\zeta_i^2 - a^2 < 0$, $\zeta_i < 0$. This means that the signs of $G_n^{(k+2)}(\zeta_i)$ and $G_n^{(k+1)}(\zeta_i)$ are opposite from where the theorem follows easily. $\square$

For Narayana polynomials the following recurrence relation holds (see [14]):

$$(n+1)N_n(x) = (2n-1)(1+x)N_{n-1}(x) - (n-2)(x-1)^2 N_{n-2}(x) \tag{3}$$

Remark 21 It is shown in paper [9] (see part 1) of Corollary 7 there) that the roots of the Narayana polynomials N_n and N_{n-1} interlace, i.e. satisfy the first interlacing property.

Theorem 22 *The zeros of the Narayana polynomials N_n and N_{n-2} satisfy the second interlacing property.*

Indeed, it suffices to show this for the roots greater than 1, for the others it will follow from the self-reciprocity of Narayana polynomials. The claimed property results from equation (3) – if $\xi > 1$ is the greatest root of N_{n-2}, then $N_{n-1}(\xi) < 0$ (see Remark 21). By equality (3) the signs of $N_n(\xi)$ and $N_{n-1}(\xi)$ are the same. As $N_{n-1}(\xi) < 0$, one has $N_n(\xi) < 0$, i.e. ξ is greater than the greatest but one root of N_n. In the same way the second interlacing property is proved for the other roots of N_{n-2} which are > 1. $\square$

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